

COMBINATORIAL YANG-BAXTER MAPS ARISING FROM TETRAHEDRON EQUATION

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ABSTRACT. We survey the matrix product solutions of the Yang-Baxter equation obtained recently from the tetrahedron equation. They form a family of quantum R matrices of generalized quantum groups interpolating the symmetric tensor representations of $U_q(A_{n-1}^{(1)})$ and the anti-symmetric tensor representations of $U_{-q-1}(A_{n-1}^{(1)})$. We show that at $q = 0$ they all reduce to the Yang-Baxter maps called combinatorial R , and describe the latter by explicit algorithm.

1. INTRODUCTION

Tetrahedron equation [27] is a generalization of the Yang-Baxter equation [1] and serves as a key to the integrability in three dimension (3D). Typically it has the form called $RRRR$ type or $RLLL$ type:

$$\begin{aligned}\mathcal{R}_{1,2,4}\mathcal{R}_{1,3,5}\mathcal{R}_{2,3,6}\mathcal{R}_{4,5,6} &= \mathcal{R}_{4,5,6}\mathcal{R}_{2,3,6}\mathcal{R}_{1,3,5}\mathcal{R}_{1,2,4}, \\ \mathcal{L}_{1,2,4}\mathcal{L}_{1,3,5}\mathcal{L}_{2,3,6}\mathcal{R}_{4,5,6} &= \mathcal{R}_{4,5,6}\mathcal{L}_{2,3,6}\mathcal{L}_{1,3,5}\mathcal{L}_{1,2,4}.\end{aligned}$$

Here $\mathcal{R} \in \text{End}(F^{\otimes 3})$ and $\mathcal{L} \in \text{End}(V \otimes V \otimes F)$ for some vector spaces F and V . The above equalities hold in $\text{End}(F^{\otimes 6})$ and $\text{End}(V^{\otimes 3} \otimes F^{\otimes 3})$ respectively, and the indices specify the components on which \mathcal{R} and \mathcal{L} act nontrivially. We call the solutions \mathcal{R} and \mathcal{L} 3D R and 3D L , respectively.

The tetrahedron equations are reducible to the Yang-Baxter equation

$$S_{1,2}S_{1,3}S_{2,3} = S_{2,3}S_{1,3}S_{1,2}$$

if the spaces 4, 5, 6 are evaluated away appropriately [2]. Such reductions and the relevant quantum group aspects have been studied systematically in the recent work [22, 20, 21] by Okado, Sergeev and the author for the distinguished example of 3D R and 3D L originating in the quantized algebra of functions [15]. They correspond to the choice $V = \mathbb{C}^2$ and the q -oscillator Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$.

In this paper we first review the latest development in [21] concerning the reduction by trace. It generates 2^n solutions $S(z)$ to the Yang-Baxter equation from the n product of \mathcal{R} and \mathcal{L} . Any $S(z)$ is rational in the parameter q and the (multiplicative) spectral parameter z . Symmetry of $S(z)$ is described by *generalized quantum groups* [8, 9] which include quantum affine [5, 12] and super algebras of type A .

In the last Section 6, we supplement a new result, Theorem 6. It shows that at $q = 0$ each $S(z)$ yields a *combinatorial R* , a certain bijection between finite sets satisfying the Yang-Baxter equation. We describe it by explicit combinatorial algorithm generalizing [23, 10].

The notion of combinatorial R originates in the crystal base theory, a theory of quantum groups at $q = 0$ [16]. The motivation for $q = 0$ further goes back to Baxter's corner transfer matrix method [1, 4], where it corresponds to the low temperature limit manifesting fascinating combinatorial features of Yang-Baxter integrable lattice models. It has numerous applications including generalized Kostka-Foulkes polynomials, Fermionic formulas of affine Lie algebra characters, integrable cellular automata in one dimension and so forth. See for example [14, 23, 7, 24, 11, 17] and reference therein. Combinatorial R 's form most systematic examples of set-theoretical solutions to the Yang-Baxter equation (Yang-Baxter maps) [6, 26] arising from the representation theory of quantum groups.

In this paper the 3D R and the 3D L will mainly serve as the constituent of the $S(z)$ which tends to the combinatorial R at $q = 0$. However they possess a decent combinatorial aspect by themselves as pointed out in [19, eq.(2.41)] for the 3D R . In fact their limits (39) define the maps

$$\lim_{q \rightarrow 0} \mathcal{R} : \begin{pmatrix} i \\ j \\ k \end{pmatrix} \mapsto \begin{pmatrix} j + \max(i - k, 0) \\ \min(i, k) \\ j + \max(k - i, 0) \end{pmatrix}, \quad \lim_{q \rightarrow 0} \mathcal{L} : \begin{pmatrix} i \\ j \\ k \end{pmatrix} \mapsto \begin{pmatrix} j + \max(i - j - k, 0) \\ \min(i, k + j) \\ \max(k + j - i, 0) \end{pmatrix}$$

on $(\mathbb{Z}_{\geq 0})^3$ and on $\{0, 1\} \times \{0, 1\} \times \mathbb{Z}_{\geq 0}$, respectively. The tetrahedron equations survive the limit nontrivially as the *combinatorial tetrahedron equations*, e.g.,

$$\begin{array}{ccccccc} & & \mathcal{R}_{4,5,6} & \nearrow & 261344 & \xrightarrow{\mathcal{R}_{2,3,6}} & 234341 & \xrightarrow{\mathcal{R}_{1,3,5}} & 432361 & \xrightarrow{\mathcal{R}_{1,2,4}} & 432361 \\ 261435 & & & \searrow & & & & & & & \\ & & \mathcal{R}_{1,2,4} & \searrow & 621835 & \xrightarrow{\mathcal{R}_{1,3,5}} & 423815 & \xrightarrow{\mathcal{R}_{2,3,6}} & 432816 & \xrightarrow{\mathcal{R}_{4,5,6}} & 432361 \\ & & & & & & & & & & \\ & & \mathcal{R}_{4,5,6} & \nearrow & 011344 & \xrightarrow{\mathcal{L}_{2,3,6}} & 011344 & \xrightarrow{\mathcal{L}_{1,3,5}} & 110354 & \xrightarrow{\mathcal{L}_{1,2,4}} & 110354 \\ 011435 & & & \searrow & & & & & & & \\ & & \mathcal{L}_{1,2,4} & \searrow & 101535 & \xrightarrow{\mathcal{L}_{1,3,5}} & 101535 & \xrightarrow{\mathcal{L}_{2,3,6}} & 110536 & \xrightarrow{\mathcal{R}_{4,5,6}} & 110354 \end{array}$$

They constitute the local relations responsible for the Yang-Baxter equation of the combinatorial R in Corollary 9.

The layout of the paper is as follows. In Section 2 we recall the definition of the 3D R and 3D L . In Section 3 tetrahedron equations of type $RRRR$ and $LLLL$ are given with their generalization to n -layer case. In Section 4 the 2^n family of solutions $S(z)$ to the Yang-Baxter equation are constructed by applying the trace reduction. In Section 5 generalized quantum group symmetry of $S(z)$ is explained. Section 6 contains the main Theorem 6, which describes the combinatorial R arising from $S(z)$ at $q = 0$ in terms of explicit combinatorial algorithm.

Throughout the paper we assume that q is not a root of unity and use the notations:

$$(z; q)_m = \prod_{k=1}^m (1 - zq^{k-1}), \quad (q)_m = (q; q)_m, \quad \binom{m}{k}_q = \frac{(q)_m}{(q)_k (q)_{m-k}}.$$

2. 3D R AND 3D L

Let F be a Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ and $\mathbf{a}^\pm, \mathbf{k} \in \text{End}(F)$ be the operators on it called q -oscillators:

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1 - q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = q^m|m\rangle. \quad (1)$$

They satisfy the relations

$$\mathbf{k}\mathbf{a}^\pm = q^{\pm 1}\mathbf{a}^\pm\mathbf{k}, \quad \mathbf{a}^+\mathbf{a}^- = 1 - \mathbf{k}^2, \quad \mathbf{a}^-\mathbf{a}^+ = 1 - q^2\mathbf{k}^2. \quad (2)$$

We define a three dimensional R operator, 3D R for short, $\mathcal{R} \in \text{End}(F^{\otimes 3})$ by

$$\mathcal{R}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c \geq 0} \mathcal{R}_{i,j,k}^{a,b,c} |a\rangle \otimes |b\rangle \otimes |c\rangle, \quad (3)$$

where several formulas are known for the matrix element:

$$\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda+\mu=b} (-1)^\lambda q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \begin{pmatrix} i \\ \mu \end{pmatrix}_{q^2} \begin{pmatrix} j \\ \lambda \end{pmatrix}_{q^2}, \quad (4)$$

$$= \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda+\mu=b} (-1)^\lambda q^{ik+b+\lambda(c-a)+\mu(\mu-i-k-1)} \begin{pmatrix} i \\ \mu \end{pmatrix}_{q^2} \begin{pmatrix} \lambda+a \\ a \end{pmatrix}_{q^2}, \quad (5)$$

$$= \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} q^{ik+b} \oint \frac{du}{2\pi i u^{b+1}} \frac{(-q^{2+a+c}u; q^2)_\infty (-q^{-i-k}u; q^2)_\infty}{(-q^{a-c}u; q^2)_\infty (-q^{c-a}u; q^2)_\infty}. \quad (6)$$

where $\delta_k^j = \delta_{j,k}$ just to save the space. The sum (4) is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ satisfying $\lambda + \mu = b$, $\mu \leq i$ and $\lambda \leq j$. The sum (5) is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ satisfying $\lambda + \mu = b$ and $\mu \leq i$. The integral (6) encircles $u = 0$ anti-clockwise so as to pick the coefficient of u^b . Derivation of these formulas can be found in [19, Th.2] for (4), [18, Sec.4] for (5) and [25] for (6). The 3D R can also be expressed as a collection of operators on the third component. For example (4) yields

$$\mathcal{R}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b \geq 0} |a\rangle \otimes |b\rangle \otimes \mathcal{R}_{i,j}^{a,b} |k\rangle, \quad \mathcal{R}_{i,j}^{a,b} \in \text{End}(F), \quad (7)$$

$$\mathcal{R}_{i,j}^{a,b} = \delta_{i+j}^{a+b} \sum_{\lambda+\mu=b} (-1)^\lambda q^{\lambda+\mu^2-ib} \begin{pmatrix} i \\ \mu \end{pmatrix}_{q^2} \begin{pmatrix} j \\ \lambda \end{pmatrix}_{q^2} (\mathbf{a}^-)^\mu (\mathbf{a}^+)^{j-\lambda} \mathbf{k}^{i+\lambda-\mu}, \quad (8)$$

where the sum is taken under the same condition as in (4), which guarantees that the powers of q -oscillators are nonnegative.

The 3D R was first obtained as the intertwiner of the quantized coordinate ring $A_q(sl_3)$ [15]¹. It was found later also from a quantum geometry consideration in a different gauge [2]. They were shown to be the same object in [19, eq.(2.29)]. See also [20, App. A] and [18, Sec. 4] for the recursion relations characterizing \mathcal{R} and useful corollaries. Here we note the properties [19]

$$\mathcal{R} = \mathcal{R}^{-1}, \quad \mathcal{R}_{i,j,k}^{a,b,c} = \mathcal{R}_{k,j,i}^{c,b,a} \in q^\xi \mathbb{Z}[q^2], \quad \mathcal{R}_{i,j,k}^{a,b,c} = \frac{(q^2)_i (q^2)_j (q^2)_k}{(q^2)_a (q^2)_b (q^2)_c} \mathcal{R}_{a,b,c}^{i,j,k}, \quad (9)$$

where $\xi = 0, 1$ is specified by $\xi \equiv (a-j)(c-j) \pmod{2}$.

¹ The formula for it on p194 in [15] contains a misprint unfortunately. The formula (4) here is a correction of it.

Example 1. The following is the list of all the nonzero $\mathcal{R}_{3,1,2}^{a,b,c}$.

$$\begin{aligned}\mathcal{R}_{3,1,2}^{1,3,0} &= -q^2(1-q^4)(1-q^6), & \mathcal{R}_{3,1,2}^{2,2,1} &= (1+q^2)(1-q^6)(1-q^2-q^6), \\ \mathcal{R}_{3,1,2}^{1,3,0} &= q^6, & \mathcal{R}_{3,1,2}^{3,1,2} &= -q^2(-1-q^2+q^6+q^8+q^{10}).\end{aligned}$$

We see $\lim_{q \rightarrow 0} \mathcal{R}_{3,1,2}^{a,b,c} = \delta_2^a \delta_2^b \delta_1^c$ in agreement with (39) with $\epsilon = 0$.

The following is the list of all the nonzero $\mathcal{R}_{3,1}^{a,b}$.

$$\begin{aligned}\mathcal{R}_{3,1}^{1,3} &= (\mathbf{a}^-)^3 \mathbf{a}^+ - q^{-4}(1+q^2+q^4)(\mathbf{a}^-)^2 \mathbf{k}^2, & \mathcal{R}_{3,1}^{4,0} &= \mathbf{a}^+ \mathbf{k}^3, \\ \mathcal{R}_{3,1}^{3,1} &= q^{-2}(1+q^2+q^4) \mathbf{a}^- \mathbf{a}^+ \mathbf{k}^2 - q^{-2} \mathbf{k}^4, & \mathcal{R}_{3,1}^{0,4} &= -q^{-2}(\mathbf{a}^-)^3 \mathbf{k}, \\ \mathcal{R}_{3,1}^{2,2} &= q^{-4}(1+q^2+q^4)(q^2(\mathbf{a}^-)^2 \mathbf{a}^+ \mathbf{k} - \mathbf{a}^- \mathbf{k}^3).\end{aligned}$$

A part of them will be used in Example 2.

Let us proceed to the 3D L [2]. Set $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$. We define a three dimensional L operator, 3D L for short, $\mathcal{L} \in \text{End}(V^{\otimes 2} \otimes F)$ by a format parallel with (7):

$$\mathcal{L}(v_\alpha \otimes v_\beta \otimes |m\rangle) = \sum_{\gamma, \delta} v_\gamma \otimes v_\delta \otimes \mathcal{L}_{\alpha, \beta}^{\gamma, \delta} |m\rangle, \quad (10)$$

where $\mathcal{L}_{\alpha, \beta}^{\gamma, \delta} \in \text{End}(F)$ are zero except the following six cases:

$$\mathcal{L}_{0,0}^{0,0} = \mathcal{L}_{1,1}^{1,1} = 1, \quad \mathcal{L}_{0,1}^{0,1} = -q\mathbf{k}, \quad \mathcal{L}_{1,0}^{1,0} = \mathbf{k}, \quad \mathcal{L}_{1,0}^{0,1} = \mathbf{a}^-, \quad \mathcal{L}_{0,1}^{1,0} = \mathbf{a}^+. \quad (11)$$

Thus \mathcal{L} may be regarded as defining a six-vertex model [1] whose Boltzmann weights take values in the q -oscillators. One may also write (10) like (3) as

$$\begin{aligned}\mathcal{L}(v_\alpha \otimes v_\beta \otimes |m\rangle) &= \sum_{\gamma, \delta, j} \mathcal{L}_{\alpha, \beta, m}^{\gamma, \delta, j} v_\gamma \otimes v_\delta \otimes |j\rangle, \\ \mathcal{L}_{0,0,m}^{0,0,j} &= \mathcal{L}_{1,1,m}^{1,1,j} = \delta_m^j, & \mathcal{L}_{0,1,m}^{0,1,j} &= -\delta_m^j q^{m+1}, & \mathcal{L}_{1,0,m}^{1,0,j} &= \delta_m^j q^m, \\ \mathcal{L}_{1,0,m}^{0,1,j} &= \delta_{m-1}^j (1 - q^{2m}), & \mathcal{L}_{0,1,m}^{1,0,j} &= \delta_{m+1}^j.\end{aligned} \quad (12)$$

The other $\mathcal{L}_{\alpha, \beta, m}^{\gamma, \delta, j}$ are zero.

We assign a solid arrow to F and a dotted arrow to V , and depict the matrix elements of 3D R and 3D L as

$$\mathcal{R}_{i,j,k}^{a,b,c} = \begin{array}{c} \begin{array}{ccc} & b & \\ & \uparrow & \\ i & \swarrow & k \\ & \downarrow & \\ c & \leftarrow & a \\ & \downarrow & \\ & j & \end{array} \end{array} \quad \mathcal{L}_{i,j,k}^{a,b,c} = \begin{array}{c} \begin{array}{ccc} & b & \\ & \uparrow & \\ i & \cdots \swarrow & k \\ & \downarrow & \\ c & \leftarrow & a \\ & \downarrow & \\ & j & \end{array} \end{array}$$

We will also depict \mathcal{R} and \mathcal{L} by the same diagrams with no indices.

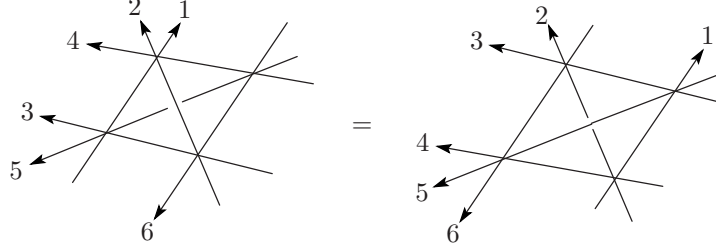
3. TETRAHEDRON EQUATION

The \mathcal{R} satisfies the tetrahedron equation of $RRRR$ type [15]

$$\mathcal{R}_{1,2,4} \mathcal{R}_{1,3,5} \mathcal{R}_{2,3,6} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{R}_{2,3,6} \mathcal{R}_{1,3,5} \mathcal{R}_{1,2,4}, \quad (13)$$

which is an equality in $\text{End}(F^{\otimes 6})$. Here $\mathcal{R}_{i,j,k}$ acts as \mathcal{R} on the i, j, k th components from the left in the tensor product $F^{\otimes 6}$, and as identity elsewhere². By denoting the F at the i th component by a solid arrow with i , (13) is depicted as follows:

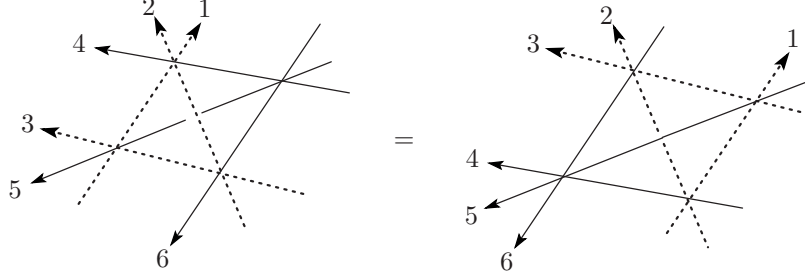
² These indices should not be confused with those specifying the matrix elements $\mathcal{R}_{i,j,k}^{a,b,c}$.



The \mathcal{L} satisfies the tetrahedron equation of $RLLL$ type [2]

$$\mathcal{L}_{1,2,4}\mathcal{L}_{1,3,5}\mathcal{L}_{2,3,6}\mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6}\mathcal{L}_{2,3,6}\mathcal{L}_{1,3,5}\mathcal{L}_{1,2,4}, \quad (14)$$

which is an equality in $\text{End}(V^{\otimes 3} \otimes F^{\otimes 3})$. The indices are assigned according to the same rule as in (13). By denoting the V at the i th component by a dotted arrow with i , (14) is depicted as follows:



Viewed as an equation on \mathcal{R} , (14) is equivalent to the intertwining relation for the irreducible representations of the quantized coordinate ring $A_q(sl_3)$ [15], [19, eq.(2.15)] in the sense that the both lead to the same solution given in (4)–(6) up to an overall normalization.

One can concatenate the tetrahedron equations to form the n -layer versions mixing the two types (13) and (14) arbitrarily. To describe them we introduce the notation unifying F, V and \mathcal{R}, \mathcal{L} .

$$W^{(\epsilon)} = \begin{cases} F, \\ V, \end{cases} \quad \mathcal{S}^{(\epsilon)} = \begin{cases} \mathcal{R}, \\ \mathcal{L}, \end{cases} \quad \mathcal{S}^{(\epsilon)}_{i,j}{}^{a,b} = \begin{cases} \mathcal{R}_{i,j}^{a,b}, \\ \mathcal{L}_{i,j}^{a,b}, \end{cases} \quad \mathcal{S}^{(\epsilon)}_{i,j,k}{}^{a,b,c} = \begin{cases} \mathcal{R}_{i,j,k}^{a,b,c} & (\epsilon = 0), \\ \mathcal{L}_{i,j,k}^{a,b,c} & (\epsilon = 1). \end{cases} \quad (15)$$

Note that

$$\mathcal{S}^{(\epsilon)}_{i,j,k}{}^{a,b,c} = 0 \quad \text{unless} \quad (a+b, b+c) = (i+j, j+k). \quad (16)$$

Now (13) and (14) are written as

$$\mathcal{S}_{1,2,4}^{(\epsilon)} \mathcal{S}_{1,3,5}^{(\epsilon)} \mathcal{S}_{2,3,6}^{(\epsilon)} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{S}_{2,3,6}^{(\epsilon)} \mathcal{S}_{1,3,5}^{(\epsilon)} \mathcal{S}_{1,2,4}^{(\epsilon)} \quad (\epsilon = 0, 1) \quad (17)$$

which is an equality in $\text{End}(W^{(\epsilon)} \otimes W^{(\epsilon)} \otimes W^{(\epsilon)} \otimes F \otimes F \otimes F)$.

Let n be a positive integer. Given an arbitrary sequence $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$, we set

$$\mathcal{W} = W^{(\epsilon_1)} \otimes \dots \otimes W^{(\epsilon_n)}. \quad (18)$$

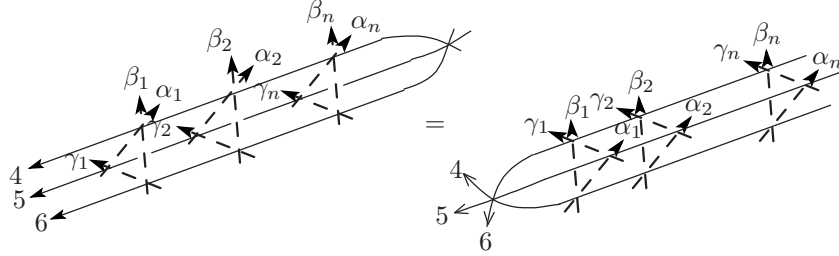
Let $W^{\alpha_i}_{(\epsilon_i)}, W^{\beta_i}_{(\epsilon_i)}, W^{\gamma_i}_{(\epsilon_i)}$ be copies of $W^{(\epsilon_i)}$, where α_i, β_i and γ_i ($i = 1, \dots, n$) are distinct labels. Replacing the spaces 1, 2, 3 by them in (17) we have

$$\mathcal{S}^{(\epsilon_i)}_{\alpha_i, \beta_i, 4} \mathcal{S}^{(\epsilon_i)}_{\alpha_i, \gamma_i, 5} \mathcal{S}^{(\epsilon_i)}_{\beta_i, \gamma_i, 6} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{S}^{(\epsilon_i)}_{\beta_i, \gamma_i, 6} \mathcal{S}^{(\epsilon_i)}_{\alpha_i, \gamma_i, 5} \mathcal{S}^{(\epsilon_i)}_{\alpha_i, \beta_i, 4}$$

for each i . Thus for any i one can let $\mathcal{R}_{4,5,6}$ penetrate $\mathcal{S}^{(\epsilon_i)}_{\alpha_i, \beta_i, 4} \mathcal{S}^{(\epsilon_i)}_{\alpha_i, \gamma_i, 5} \mathcal{S}^{(\epsilon_i)}_{\beta_i, \gamma_i, 6}$ to the left transforming it into the reverse order product $\mathcal{S}^{(\epsilon_i)}_{\beta_i, \gamma_i, 6} \mathcal{S}^{(\epsilon_i)}_{\alpha_i, \gamma_i, 5} \mathcal{S}^{(\epsilon_i)}_{\alpha_i, \beta_i, 4}$. Repeating this n times leads to

$$\begin{aligned} & (\mathcal{S}^{(\epsilon_1)}_{\alpha_1, \beta_1, 4} \mathcal{S}^{(\epsilon_1)}_{\alpha_1, \gamma_1, 5} \mathcal{S}^{(\epsilon_1)}_{\beta_1, \gamma_1, 6}) \cdots (\mathcal{S}^{(\epsilon_n)}_{\alpha_n, \beta_n, 4} \mathcal{S}^{(\epsilon_n)}_{\alpha_n, \gamma_n, 5} \mathcal{S}^{(\epsilon_n)}_{\beta_n, \gamma_n, 6}) \mathcal{R}_{4,5,6} \\ &= \mathcal{R}_{4,5,6} (\mathcal{S}^{(\epsilon_1)}_{\beta_1, \gamma_1, 6} \mathcal{S}^{(\epsilon_1)}_{\alpha_1, \gamma_1, 5} \mathcal{S}^{(\epsilon_1)}_{\alpha_1, \beta_1, 4}) \cdots (\mathcal{S}^{(\epsilon_n)}_{\beta_n, \gamma_n, 6} \mathcal{S}^{(\epsilon_n)}_{\alpha_n, \gamma_n, 5} \mathcal{S}^{(\epsilon_n)}_{\alpha_n, \beta_n, 4}). \end{aligned} \quad (19)$$

This is an equality in $\text{End}(\tilde{\mathcal{W}}^{\alpha} \otimes \tilde{\mathcal{W}}^{\beta} \otimes \tilde{\mathcal{W}}^{\gamma} \otimes \overset{4}{F} \otimes \overset{5}{F} \otimes \overset{6}{F})$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is the array of labels and $\tilde{\mathcal{W}}^{\alpha} = W^{\alpha_1}_{(\epsilon_1)} \otimes \cdots \otimes W^{\alpha_n}_{(\epsilon_n)}$. The spaces $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}$ should be understood similarly. They are just copies of \mathcal{W} in (18). The relation (19) is depicted as follows:



Here the broken arrows represent either solid or dotted arrows depending on whether the corresponding ϵ_i is 0 or 1. The vertices on the i th layer $\mathcal{S}^{(\epsilon_i)}$ should also be understood as \mathcal{R} or \mathcal{L} accordingly.

4. SOLUTION TO THE YANG-BAXTER EQUATION

One can reduce (19) to the Yang-Baxter equation involving spectral parameters. In this paper we shall only consider the reduction by trace. See [22, 21] for another reduction by using boundary vectors.

Define $\mathbf{h} \in \text{End}(F)$ by $\mathbf{h}|m\rangle = m|m\rangle$. By (16), $[x^{\mathbf{h}_4 + \mathbf{h}_5} y^{\mathbf{h}_5 + \mathbf{h}_6}, \mathcal{R}_{4,5,6}] = 0$ holds for parameters x and y , where the indices specify the spaces on which the operators act nontrivially. Multiply $\mathcal{R}_{4,5,6}^{-1} x^{\mathbf{h}_4 + \mathbf{h}_5} y^{\mathbf{h}_5 + \mathbf{h}_6} = x^{\mathbf{h}_4 + \mathbf{h}_5} y^{\mathbf{h}_5 + \mathbf{h}_6} \mathcal{R}_{4,5,6}^{-1}$ from the left to (19) and take the trace over the space $F^{\otimes 3}$ corresponding to 4, 5, 6. The result becomes the Yang-Baxter equation

$$S_{\alpha, \beta}(x) S_{\alpha, \gamma}(xy) S_{\beta, \gamma}(y) = S_{\beta, \gamma}(y) S_{\alpha, \gamma}(xy) S_{\alpha, \beta}(x) \in \text{End}(\tilde{\mathcal{W}}^{\alpha} \otimes \tilde{\mathcal{W}}^{\beta} \otimes \tilde{\mathcal{W}}^{\gamma}) \quad (20)$$

for the matrix $S_{\alpha, \beta}(z) \in \text{End}(\tilde{\mathcal{W}}^{\alpha} \otimes \tilde{\mathcal{W}}^{\beta})$ constructed as

$$S_{\alpha, \beta}(z) = \text{Tr}_3 \left(z^{\mathbf{h}_3} \mathcal{S}^{(\epsilon_1)}_{\alpha_1, \beta_1, 3} \cdots \mathcal{S}^{(\epsilon_n)}_{\alpha_n, \beta_n, 3} \right), \quad (21)$$

where 3 denotes a copy of F . To describe the matrix elements of $S_{\alpha,\beta}(z)$ we write the basis of (18) as

$$\mathcal{W} = \bigoplus_{m_1, \dots, m_n} \mathbb{C}|m_1, \dots, m_n\rangle, \quad |m_1, \dots, m_n\rangle = |m_1\rangle^{(\epsilon_1)} \otimes \dots \otimes |m_n\rangle^{(\epsilon_n)}, \quad (22)$$

$$|m\rangle^{(0)} = |m\rangle \in F \quad (m \in \mathbb{Z}_{\geq 0}), \quad |m\rangle^{(1)} = v_m \in V \quad (m \in \{0, 1\}). \quad (23)$$

The range of the indices m_i are to be understood as $\mathbb{Z}_{\geq 0}$ or $\{0, 1\}$ according to $\epsilon_i = 0$ or 1 as in (23). It will crudely be denoted by $0 \leq m_i \leq 1/\epsilon_i$. We use the shorthand $|\mathbf{m}\rangle = |m_1, \dots, m_n\rangle$ for $\mathbf{m} = (m_1, \dots, m_n)$ and write (22) as $\mathcal{W} = \bigoplus_{\mathbf{m}} \mathbb{C}|\mathbf{m}\rangle$. We set $|\mathbf{m}| = m_1 + \dots + m_n$.

Let $S(z) \in \text{End}(\mathcal{W} \otimes \mathcal{W})$ denote the solution (21) of the Yang-Baxter equation, where the inessential labels α, β are now suppressed³. Remember, however, that $S(z)$ depends on the choice $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$. We write its action as

$$S(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a}, \mathbf{b}} S(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle. \quad (24)$$

Then the matrix elements are given by

$$S(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = \text{Tr}_F \left(z^{\mathbf{h}} \mathcal{S}_{i_1, j_1}^{(\epsilon_1) a_1, b_1} \dots \mathcal{S}_{i_n, j_n}^{(\epsilon_n) a_n, b_n} \right) \quad (25)$$

$$= \sum_{c_0, \dots, c_{n-1}} z^{c_0} \mathcal{S}_{i_1, j_1, c_0}^{(\epsilon_1) a_1, b_1, c_0} \mathcal{S}_{i_2, j_2, c_0}^{(\epsilon_2) a_2, b_2, c_1} \dots \mathcal{S}_{i_n, j_n, c_{n-1}}^{(\epsilon_n) a_n, b_n, c_{n-1}}. \quad (26)$$

The operators in (25) are defined by (15), (11) and (8). From (16) it follows that

$$S(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = 0 \quad \text{unless} \quad \mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} \quad \text{and} \quad |\mathbf{a}| = |\mathbf{i}|, \quad |\mathbf{b}| = |\mathbf{j}|. \quad (27)$$

Given such $\mathbf{a}, \mathbf{b}, \mathbf{i}$ and \mathbf{j} , (16) further reduces the sums over $c_i \in \mathbb{Z}_{\geq 0}$ in (26) effectively into a *single* sum. The latter property in (27) implies the direct sum decomposition:

$$S(z) = \bigoplus_{l, m \geq 0} S_{l, m}(z), \quad S_{l, m}(z) \in \text{End}(\mathcal{W}_l \otimes \mathcal{W}_m), \quad \mathcal{W}_l = \bigoplus_{\mathbf{m}, |\mathbf{m}|=l} \mathbb{C}|\mathbf{m}\rangle \subset \mathcal{W}, \quad (28)$$

where the former sum ranges over $0 \leq l, m \leq n$ if $\epsilon_1 \dots \epsilon_n = 1$ and $l, m \in \mathbb{Z}_{\geq 0}$ otherwise. The formula (25) is depicted as

$$S(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = \text{Tr}_F \left(\begin{array}{c} \begin{array}{c} \text{Diagram showing the matrix product construction of } S(z) \text{ in terms of 3D } R \text{ and 3D } L \text{ with the auxiliary space } F. \end{array} \end{array} \right).$$

Here the broken arrows represent either solid or dotted arrows according to $\epsilon_i = 0$ or 1 at the corresponding site. Thus (25) is a matrix product construction of $S(z)$ in terms of 3D R and 3D L with the auxiliary space F .

³ The labels $\alpha, \beta \dots$ introduced for the exposition of (20) will no longer be used in the rest of the paper, and should not be confused with the indices of $S_{l, m}(z)$ in (28).

Example 2. Take $n = 3$ and $(\epsilon_1, \epsilon_2, \epsilon_3) = (1, 0, 1)$. Then one has

$$\begin{aligned} S(z)(|031\rangle \otimes |110\rangle) &= S_{031,110}^{031,110}(z)|031\rangle \otimes |110\rangle + S_{031,110}^{040,101}(z)|040\rangle \otimes |101\rangle \\ &\quad + S_{031,110}^{121,020}(z)|121\rangle \otimes |020\rangle + S_{031,110}^{130,011}(z)|130\rangle \otimes |011\rangle, \end{aligned}$$

where the matrix elements are expressed as

$$\begin{aligned} S_{031,110}^{031,110}(z) &= \text{Tr}(z^{\mathbf{h}} \mathcal{L}_{0,1}^{0,1} \mathcal{R}_{3,1}^{3,1} \mathcal{L}_{1,0}^{1,0}), & S_{031,110}^{040,101}(z) &= \text{Tr}(z^{\mathbf{h}} \mathcal{L}_{0,1}^{0,1} \mathcal{R}_{3,1}^{4,0} \mathcal{L}_{1,0}^{0,1}), \\ S_{031,110}^{121,020}(z) &= \text{Tr}(z^{\mathbf{h}} \mathcal{L}_{0,1}^{1,0} \mathcal{R}_{3,1}^{2,2} \mathcal{L}_{1,0}^{1,0}), & S_{031,110}^{130,011}(z) &= \text{Tr}(z^{\mathbf{h}} \mathcal{L}_{0,1}^{1,0} \mathcal{R}_{3,1}^{3,1} \mathcal{L}_{1,0}^{0,1}). \end{aligned}$$

Using $\mathcal{L}_{i,j}^{a,b}$ (11) and $\mathcal{R}_{i,j}^{a,b}$ in Example 1, one calculates them for instance as

$$\begin{aligned} S_{031,110}^{040,101}(z) &= \text{Tr}(z^{\mathbf{h}}(-q\mathbf{k})\mathbf{a}^+\mathbf{k}^3\mathbf{a}^-) = -q^{-2}\text{Tr}(\mathbf{k}^4 z^{\mathbf{h}}\mathbf{a}^+\mathbf{a}^-) \\ &= -q^{-2} \sum_{m \geq 0} (q^4 z)^m (1 - q^{2m}) = \frac{-q^2(1 - q^2)z}{(1 - q^4 z)(1 - q^6 z)}. \end{aligned}$$

Similar calculations lead to

$$\begin{aligned} S_{031,110}^{031,110}(z) &= \frac{q^3(q^2 - z)}{(1 - q^4 z)(1 - q^6 z)}, & S_{031,110}^{121,020}(z) &= \frac{-q^2(1 - q^6)(q^2 - z)z}{(1 - q^2 z)(1 - q^4 z)(1 - q^6 z)}, \\ S_{031,110}^{130,011}(z) &= \frac{-(1 - q^2)z(q^4 - z - q^2 z + q^8 z)}{(1 - q^2 z)(1 - q^4 z)(1 - q^6 z)}. \end{aligned}$$

In general $S(z)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}}$ is a *rational* function of q and z .

Example 3. For $0 \leq a, b, i, j \leq 1$, $\mathcal{R}_{i,j}^{a,b}$ (8) and $\mathcal{L}_{i,j}^{a,b}$ (11) are the same except $\mathcal{R}_{1,1}^{1,1} = \mathbf{a}^- \mathbf{a}^+ - \mathbf{k}^2$ and $\mathcal{L}_{1,1}^{1,1} = 1$. This implies that $S(z)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}}$ with $(a_\alpha, b_\alpha, i_\alpha, j_\alpha) = (1, 1, 1, 1)$ depends on $\epsilon_\alpha = 0, 1$. The following table shows such examples, in which the case $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (0, 0, 0, 0)$ is omitted since the expression is too bulky.

$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$	$(0, 1, 0, 1)$	$(0, 1, 0, 0)$	$(0, 0, 0, 1)$
$S_{0121,1101}^{1111,0111}(z)$	$\frac{(1-q^4)z}{(1-qz)(1-q^3z)}$	$\frac{(1-q^4)z(1-q^2-q^4+q^3z)}{(1-qz)(1-q^3z)(1-q^5z)}$	$-\frac{q(1-q^4)z(q-z-q^2z+q^4z)}{(1-qz)(1-q^3z)(1-q^5z)}$

5. GENERALIZED QUANTUM GROUP SYMMETRY

The $S(z)$ constructed in the previous section possesses the generalized quantum group symmetry. Recall that $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$ is an arbitrary sequence. Set

$$q_i = (-1)^{\epsilon_i} q^{1-2\epsilon_i}, \quad D_{i,j} = \prod_{k \in \{i, i+1\} \cap \{j, j+1\}} (q_k)^{2\delta_{i,j}-1} \quad (i, j \in \mathbb{Z}_n). \quad (29)$$

We introduce the $\mathbb{C}(q)$ -algebra $\mathcal{U}_A = \mathcal{U}_A(\epsilon_1, \dots, \epsilon_n)$ generated by $e_i, f_i, k_i^{\pm 1}$ ($i \in \mathbb{Z}_n$) obeying the relations

$$\begin{aligned} k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0, \\ k_i e_j &= D_{i,j} e_j k_i, \quad k_i f_j = D_{i,j}^{-1} f_j k_i, \quad [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}}. \end{aligned} \quad (30)$$

We endow it with the Hopf algebra structure with coproduct Δ , counit ε and antipode \mathcal{S} as follows:

$$\begin{aligned} \Delta k_i^{\pm 1} &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i, \\ \varepsilon(k_i) &= 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \mathcal{S}(k_i^{\pm 1}) = k_i^{\mp 1}, \quad \mathcal{S}(e_i) = -e_i k_i^{-1}, \quad \mathcal{S}(f_i) = -k_i f_i. \end{aligned} \quad (31)$$

With a supplement of appropriate Serre relations, the homogeneous cases $\epsilon_1 = \dots = \epsilon_n$ are identified with the quantum affine algebras [5, 12] as

$$\mathcal{U}_A(0, \dots, 0) = U_q(A_{n-1}^{(1)}), \quad \mathcal{U}_A(1, \dots, 1) = U_{-q^{-1}}(A_{n-1}^{(1)}). \quad (32)$$

In general $\mathcal{U}_A(\epsilon_1, \dots, \epsilon_n)$ is an example of generalized quantum groups [8, 9] including an affinization of quantum super algebra $sl_q(\kappa, n - \kappa)$. See [21, Sec.3.3] for more detail.

For the space \mathcal{W}_l (28) and a parameter x , the following map $\pi_x^{(l)} : \mathcal{U}_A(\epsilon_1, \dots, \epsilon_n) \rightarrow \text{End}(\mathcal{W}_l)$ gives an irreducible finite dimensional representation⁴

$$\begin{aligned} e_i |\mathbf{m}\rangle &= x^{\delta_{i,0}} [m_i] |\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, \\ f_i |\mathbf{m}\rangle &= x^{-\delta_{i,0}} [m_{i+1}] |\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, \\ k_i |\mathbf{m}\rangle &= (q_i)^{-m_i} (q_{i+1})^{m_{i+1}} |\mathbf{m}\rangle, \end{aligned} \quad (33)$$

where $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $\mathbf{e}_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^n$. The vectors $|\mathbf{m}'\rangle = |m'_1, \dots, m'_n\rangle$ on the rhs of (33) are to be understood as zero unless $0 \leq m'_i \leq 1/\epsilon_i$ for all $1 \leq i \leq n$. In the homogeneous case, the representation $\pi_x^{(l)}$ is equivalent to

degree l symmetric tensor rep. of $U_q(A_{n-1}^{(1)})$ for $\epsilon_1 = \dots = \epsilon_n = 0$,

degree l anti-symmetric tensor rep. of $U_{-q^{-1}}(A_{n-1}^{(1)})$ for $\epsilon_1 = \dots = \epsilon_n = 1$.

Let Δ' denote the opposite (i.e., the left and the right components interchanged) coproduct of Δ in (31).

Theorem 4. ([21, Th.5.1]) *For any $l, m \in \mathbb{Z}_{\geq 0}$, the following commutativity holds:*

$$\Delta'(g) S_{l,m}(z) = S_{l,m}(z) \Delta(g) \quad \forall g \in \mathcal{U}_A(\epsilon_1, \dots, \epsilon_n),$$

where $\Delta(g)$ and $\Delta'(g)$ stand for the tensor product representations $(\pi_x^{(l)} \otimes \pi_y^{(m)}) \Delta(g)$ and $(\pi_x^{(l)} \otimes \pi_y^{(m)}) \Delta'(g)$ of (33) with $z = x/y$.

If $\mathcal{W}_l \otimes \mathcal{W}_m$ is irreducible, Theorem 4 characterizes $S_{l,m}(z)$ up to an overall scalar. Therefore $S_{l,m}(z)$ is identified with the *quantum R matrix* in the sense of [12] associated with $\mathcal{U}_A(\epsilon_1, \dots, \epsilon_n)$ -module $\mathcal{W}_l \otimes \mathcal{W}_m$. Although we expect that $\mathcal{W}_l \otimes \mathcal{W}_m$ is irreducible for arbitrary $(\epsilon_1, \dots, \epsilon_n)$, it has hitherto been proved rigorously only for $(\epsilon_1, \dots, \epsilon_n)$ of the form $(1^\kappa, 0^{n-\kappa})$ with $0 \leq \kappa \leq n$ [21]. Anyway the family $S_{l,m}(z)$ (25) interpolates the quantum R matrices for the symmetric tensor representations of $U_q(A_{n-1}^{(1)})$ and the anti-symmetric tensor representations of $U_{-q^{-1}}(A_{n-1}^{(1)})$ as the two extreme cases $\kappa = 0$ and n . In [21, Prop.2.1], it was also shown that $S_{l,m}(z)$'s associated with $(\epsilon_1, \dots, \epsilon_n)$ and $(\epsilon'_1, \dots, \epsilon'_n)$ are connected by a similarity transformation if the two sequences are permutations of each other. Thus one can claim that *all* the $S_{l,m}(z)$ (25) are equivalent to the quantum R matrices of some generalized quantum group.

6. COMBINATORIAL R

In this section we study $S_{l,m}(z)$ (28) at $q = 0$. Let $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$ be an arbitrary sequence and introduce the *crystal*

$$B_l = \{\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{Z}_{\geq 0})^n \mid |\mathbf{a}| = l, 0 \leq a_i \leq 1/\epsilon_i (1 \leq i \leq n)\}, \quad (34)$$

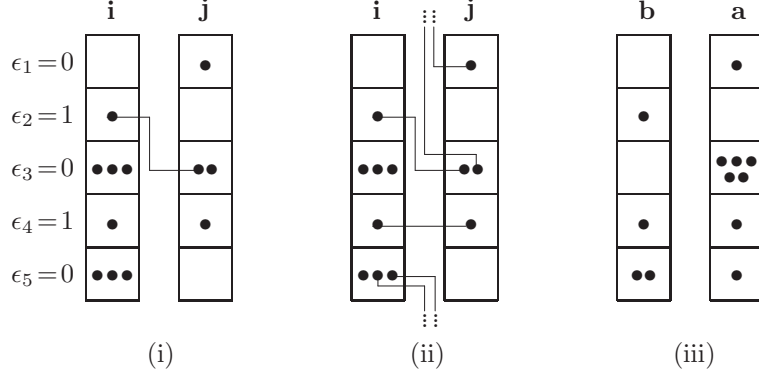
⁴ Image $\pi_x^{(l)}(g)$ is denoted by g for simplicity.

which is a finite labeling set of the basis of \mathcal{W}_l (28). We identify $\mathbf{a} = (a_1, \dots, a_n) \in B_l$ with the depth n column shape tableau containing a_i dots in the i th box from the top ($1 \leq i \leq n$). See the diagrams given below. Call the dots in the i th box *bosonic* if $\epsilon_i = 0$ and *fermionic* if $\epsilon_i = 1$. Thus there are l dots in the tableau in total among which $\epsilon_1 a_1 + \dots + \epsilon_n a_n$ are fermionic and the rest are bosonic.

We are going to define a map $R = R_{l,m} : B_l \otimes B_m \rightarrow B_m \otimes B_l$ and a function $H = H_{l,m} : B_l \otimes B_m \rightarrow \mathbb{Z}_{\geq 0}$ by combinatorial algorithm, where \otimes may just be understood as a product of sets. Thus for a given pair of tableaux $\mathbf{i} \otimes \mathbf{j} \in B_l \otimes B_m$, we are to specify the right hand sides of

$$R(\mathbf{i} \otimes \mathbf{j}) = \mathbf{b} \otimes \mathbf{a} \in B_m \otimes B_l, \quad H(\mathbf{i} \otimes \mathbf{j}) = w \in \mathbb{Z}_{\geq 0}. \quad (35)$$

For $l \geq m$, it is done by the algorithm (i)–(iii) given below:



- (i) Choose a dot, say d , in \mathbf{j} and connect it to a dot d' in \mathbf{i} to form a pair. If d is bosonic (resp. fermionic), d' should be the lowest one among those located strictly higher (resp. not strictly lower) than d . If there is no such dot, take d' to be the lowest one in \mathbf{i} . Such a pair is called *winding*. The lines pairing the dots are called *H-lines*.
- (ii) Repeat (i) for yet unpaired dots until all dots in \mathbf{j} are paired to some dots in \mathbf{i} .
- (iii) Move the $l - m$ unpaired dots in \mathbf{i} horizontally to \mathbf{j} . The resulting tableaux define $\mathbf{b} \otimes \mathbf{a}$. w is the winding number (number of winding pairs).

The above example is for $n = 5$, $(\epsilon_1, \dots, \epsilon_5) = (0, 1, 0, 1, 0)$, $B_l \otimes B_m = B_8 \otimes B_4$ and shows

$$R(01313 \otimes 10210) = 01012 \otimes 10511, \quad H(01313 \otimes 10210) = 2.$$

Remark 5.

- (1) In (i) and (ii), the *H-lines* depend on the order of choosing the dots from \mathbf{j} . However, the final result of $\mathbf{b} \otimes \mathbf{a}$ and w can be shown to be independent of it.
- (2) The *H-lines* in the winding case are naturally interpreted as going up *periodically* along the tableaux.
- (3) The condition of being bosonic or fermionic in (i) only refers to d and does not concern d' .
- (4) When $l = m$, R is trivial in that $R(\mathbf{i} \otimes \mathbf{j}) = \mathbf{i} \otimes \mathbf{j}$, but $H(\mathbf{i} \otimes \mathbf{j})$ remains nontrivial.

- (5) The algorithm also specifies the number c_t of the H -lines passing through the border between the t th and the $(t+1)$ th components in the tableaux \mathbf{i} and \mathbf{j} for $t \in \mathbb{Z}_n$. The winding number is $c_0 = c_n$. For instance in the above diagram (ii), we see $(c_1, \dots, c_5) = (1, 2, 0, 0, 2)$. They satisfy the piecewise linear relations:

$$\begin{array}{ll} \epsilon_t = 0 \text{ case} & \epsilon_t = 1 \text{ case} \\ \left\{ \begin{array}{l} a_t = j_t + (i_t - c_t)_+, \\ b_t = \min(i_t, c_t), \\ c_{t-1} = j_t + (c_t - i_t)_+, \end{array} \right. & \left\{ \begin{array}{l} a_t = j_t + (i_t - j_t - c_t)_+, \\ b_t = \min(i_t, c_t + j_t), \\ c_{t-1} = (j_t + c_t - i_t)_+, \end{array} \right. \end{array} \quad (36)$$

where $t \in \mathbb{Z}_n$ and $(x)_+ = \max(x, 0)$. Given \mathbf{i} and \mathbf{j} , one may regard the last rows in (36) as a closed system of piecewise linear equations on $c_1, \dots, c_n = c_0$ whose solution determines \mathbf{a} and \mathbf{b} via the first and the second rows. We will argue the uniqueness of the solution in the proof of Theorem 6.

For $l < m$, the algorithm is replaced by the following (i)'–(iii)':

- (i)' Choose a dot, say d , in \mathbf{i} and connect it to a dot d' in \mathbf{j} to form a pair. If d is bosonic (resp. fermionic), d' should be the highest one among those located strictly lower (resp. not strictly higher) than d . If there is no such dot, take d' to be the highest one in \mathbf{j} . Such a pair is called winding.
- (ii)' Repeat (i)' for yet unpaired dots until all dots in \mathbf{i} are paired to some dots in \mathbf{j} .
- (iii)' Move the $m-l$ unpaired dots in \mathbf{j} horizontally to \mathbf{i} . The resulting tableaux define $\mathbf{b} \otimes \mathbf{a}$. w is the winding number.

Analogue of Remark 5 apply to (i)'–(iii)' as well. It can be shown that $R_{l,m}R_{m,l} = \text{id}_{B_m \otimes B_l}$. Thus R is a *bijection*. By the definition $H_{l,m}(\mathbf{i} \otimes \mathbf{j}) = H_{m,l}(\mathbf{b} \otimes \mathbf{a})$ holds when $R_{l,m}(\mathbf{i} \otimes \mathbf{j}) = \mathbf{b} \otimes \mathbf{a}$ or equivalently $R_{m,l}(\mathbf{b} \otimes \mathbf{a}) = \mathbf{i} \otimes \mathbf{j}$.

The bijective map R and the function H are called (classical part of) *combinatorial R* and *energy*, respectively. For $(\epsilon_1, \dots, \epsilon_n)$ of the form $(0^r 1^{n-r})$, it was first introduced for $r = 0$ and $r = n$ as Rule 3.10 and Rule 3.11 in [23] in the framework of crystal base theory [16] of $U_q(\widehat{\mathfrak{sl}}_n)$, and later for general r in [10] based on a realization of $U_q(\mathfrak{gl}(r, n-r))$ crystals in [3]. Note that our algorithm for $r = 0$ case, i.e. $\forall \epsilon_i = 1$ coincides with [23, Rule 3.10] after reversing the conditions ‘higher’ and ‘lower’. We suppose this is due to the right relation in (32) indicating the interchange of $q = 0$ and $q = \infty$ in the two papers.

We define the matrix element of the combinatorial R as

$$R_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = \begin{cases} 1 & \text{if } R(\mathbf{i} \otimes \mathbf{j}) = \mathbf{b} \otimes \mathbf{a}, \\ 0 & \text{otherwise.} \end{cases} \quad (37)$$

Now we state the main result.

Theorem 6. *Let $S_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}(z)$ be the element (25)–(26) of $S(z) = S_{l,m}(z)$ (28). Set $R = R_{l,m}$ and $H = H_{l,m}$. Then the following equality is valid:*

$$(1-z)^{\delta_{l,m}} \lim_{q \rightarrow 0} q^{-(m-l)_+} S_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}(z) = z^{H(\mathbf{i} \otimes \mathbf{j})} R_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}. \quad (38)$$

Proof. Setting $\check{S}_{l,m}(z) = P S_{l,m}(z)$ with $P(u \otimes v) = v \otimes u$, one can show the inversion relation $\check{S}_{l,m}(z) \check{S}_{m,l}(z^{-1}) = \rho(z) \text{id}_{\mathcal{W}_m \otimes \mathcal{W}_l}$ with an explicit scalar function $\rho(z)$ by

using (2.30), (2.31), Proposition 2.1, (3.20), (3.21), Theorem 4.1, (6.10), (6.13) and (6.16) in [21]. This reduces the proof to the case $l \geq m$ on which we shall concentrate in the sequel. From [19, (2.32)], (12) and (15) we have (see also Example 1)

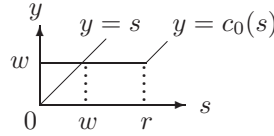
$$\lim_{q \rightarrow 0} s^{(\epsilon) a, b, c}_{i, j, k} = \begin{cases} \lim_{q \rightarrow 0} \mathcal{R}_{i, j, k}^{a, b, c} = \delta_{j+(i-k)_+}^a \delta_{\min(i, k)}^b \delta_{j+(k-i)_+}^c & (\epsilon = 0), \\ \lim_{q \rightarrow 0} \mathcal{L}_{i, j, k}^{a, b, c} = \delta_{j+(i-j-k)_+}^a \delta_{\min(i, k+j)}^b \delta_{(j+k-i)_+}^c & (\epsilon = 1), \end{cases} \quad (39)$$

which also satisfies (16). This is non-vanishing exactly when (36) is satisfied after the replacement $(\epsilon, a, b, c, i, j, k) \rightarrow (\epsilon_t, a_t, b_t, c_{t-1}, i_t, j_t, c_t)$. Therefore substitution of (39) into (26) leads to

$$\lim_{q \rightarrow 0} S_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}', \mathbf{b}'}(z) = \sum_{c_n \geq 0} \delta_{\mathbf{a}(c_n)}^{\mathbf{a}'} \delta_{\mathbf{b}(c_n)}^{\mathbf{b}'} \delta_{c_0(c_n)}^{c_n} z^{c_n}, \quad (40)$$

where $\mathbf{a}(c_n) = (a_1, \dots, a_n)$, $\mathbf{b}(c_n) = (b_1, \dots, b_n)$ and $c_0(c_n) = c_0$ are determined from c_n and \mathbf{i}, \mathbf{j} uniquely by (36) *without* the constraint $c_0 = c_n$. The origin of the factor $\delta_{c_0(c_n)}^{c_n}$ is the ‘periodic boundary condition’ implied by the trace in (25)–(26). Thus the proof is reduced to the existence and the uniqueness problem of the solution to the equation $c_0(c_n) = c_0$ on c_n .

First we assume $l > m$. Then there uniquely exists the integer $w \geq 0$ such that $w = c_0(w)$. In fact such w is given by $w = c_0(0)$. To see this note that $c_0(s+1) = c_0(s)$ or $c_0(s) + 1$ for any s because of $(x+1)_+ = (x)_+ + 1$. Let r be the smallest non-negative integer such that $c_0(r) = w$ and $c_0(r+1) = w+1$. From (36) this can happen, either for $\epsilon_t = 0$ or 1, only if $c_{t-1} = c_t + j_t - i_t$ for all $1 \leq t \leq n$. Then $w = c_0(r)$ implies $w = r + |\mathbf{j}| - |\mathbf{i}| = r + m - l < r$. Thus the unique existence of the solution to $w = c_0(w)$ is obvious from the following graph.



Now (40) reduces to the single term $\lim_{q \rightarrow 0} S_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}', \mathbf{b}'}(z) = \delta_{\mathbf{a}(w)}^{\mathbf{a}'} \delta_{\mathbf{b}(w)}^{\mathbf{b}'} z^w$, where w is the unique solution of $w = c_0(w)$. It is equal to the energy $H(\mathbf{i} \otimes \mathbf{j})$ due to Remark 5 (5), which also tells that $R(\mathbf{i} \otimes \mathbf{j}) = \mathbf{b}(w) \otimes \mathbf{a}(w)$. Therefore (38) holds.

Next we consider the case $l = m$. Again we have (40) with (36). The sum of the first row of (36) over $1 \leq t \leq n$ leads to $|\mathbf{a}| = |\mathbf{j}| + f$ with $f \geq 0$. Due to $|\mathbf{a}| = l = m = |\mathbf{j}|$, $f = 0$ must hold implying that $\mathbf{b}(c_n) \otimes \mathbf{a}(c_n) = \mathbf{i} \otimes \mathbf{j}$. As for the solution to $c_n = c_0(c_n)$, the previous argument tells that it holds for *all* $c_n \geq w = c_0(0) = H(\mathbf{i} \otimes \mathbf{j})$. Thus the right hand side of (40) becomes $\delta_{\mathbf{j}}^{\mathbf{a}'} \delta_{\mathbf{i}}^{\mathbf{b}'} \sum_{c_n \geq H(\mathbf{i} \otimes \mathbf{j})} z^{c_n}$ in agreement with (38). \square

Example 7. Taking the limit $q \rightarrow 0$ in Example 2 one has

$$\lim_{q \rightarrow 0} S(z)(|031\rangle \otimes |110\rangle) = z^2 |130\rangle \otimes |011\rangle$$

for $S(z) = S_{4,2}(z)$. This agrees with the combinatorial R^5 and the energy

$$R(031 \otimes 110) = 011 \otimes 130, \quad H(031 \otimes 110) = 2.$$

Example 8. Another check of (38), where the last line is due to Example 3.

⁵ The reason for $011 \otimes 130$ rather than $130 \otimes 011$ is due to the opposite arrangement of \mathbf{a} and \mathbf{b} in the definitions (24) and (37).

$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$	$(0,1,0,1)$	$(0,1,0,0)$	$(0,0,0,1)$
$R(0121 \otimes 1101)$	$0111 \otimes 1111$	$0111 \otimes 1111$	$0021 \otimes 1201$
$H(0121 \otimes 1101)$	1	1	2
$\lim_{q \rightarrow 0} S_{0121,1101}^{1111,0111}(z)$	z	z	0

Let us describe the Yang-Baxter equation satisfied by the combinatorial R as a corollary of (20). In order to properly treat the spectral parameter we introduce the *affine* crystal

$$\text{Aff}(B_l) = \{\mathbf{a}[d] \mid \mathbf{a} \in B_l, d \in \mathbb{Z}\}.$$

It allows us to unify the classical part of the combinatorial R and the energy H in (35) in the (full) combinatorial R $\mathcal{R} = \mathcal{R}_{l,m} : \text{Aff}(B_l) \otimes \text{Aff}(B_m) \rightarrow \text{Aff}(B_m) \otimes \text{Aff}(B_l)$ as

$$\mathcal{R}(\mathbf{i}[d] \otimes \mathbf{j}[e]) = \mathbf{b}[e - H(\mathbf{i} \otimes \mathbf{j})] \otimes \mathbf{a}[d + H(\mathbf{i} \otimes \mathbf{j})],$$

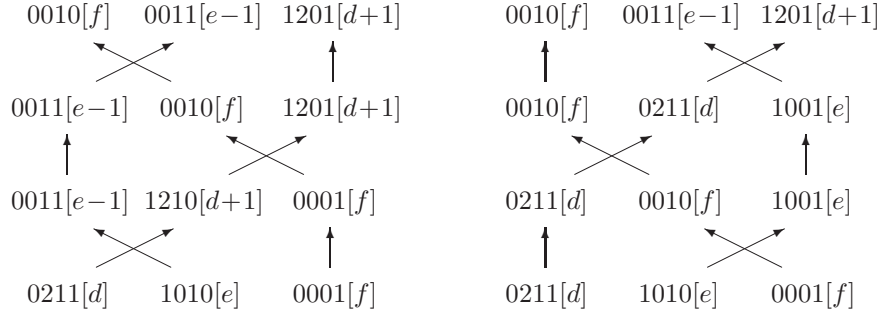
where $\mathbf{b} \otimes \mathbf{a}$ is specified by $\mathbf{b} \otimes \mathbf{a} = R(\mathbf{i} \otimes \mathbf{j})$.

Corollary 9. *The combinatorial R satisfies the Yang-Baxter equation*

$$(\mathcal{R}_{l,m} \otimes 1)(1 \otimes \mathcal{R}_{k,m})(\mathcal{R}_{k,l} \otimes 1) = (1 \otimes \mathcal{R}_{k,l})(\mathcal{R}_{k,m} \otimes 1)(1 \otimes \mathcal{R}_{l,m})$$

as maps $\text{Aff}(B_k) \otimes \text{Aff}(B_l) \otimes \text{Aff}(B_m) \rightarrow \text{Aff}(B_m) \otimes \text{Aff}(B_l) \otimes \text{Aff}(B_k)$.

Example 10. Let $n = 4$ and $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 0, 1, 0)$. We apply the two sides of Corollary 9 on the element from $\text{Aff}(B_4) \otimes \text{Aff}(B_2) \otimes \text{Aff}(B_1)$ in the bottom line.



At the top line the two sides coincide, confirming the Yang-Baxter equation.

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